

---

# Invisible Singularities in Complex Algebraic Geometry

Maurício Corrêa

Università degli Studi di Bari Aldo Moro

Joint work with János Kollár, Stefan Schreieder and Botong Wang

UCLA Birational Geometry Seminar    16 October 2026

# Zariski's original paper (1971)

---

BULLETIN OF THE  
AMERICAN MATHEMATICAL SOCIETY  
Volume 77, Number 4, July 1971

## SOME OPEN QUESTIONS IN THE THEORY OF SINGULARITIES<sup>1</sup>

BY OSCAR ZARISKI

**ABSTRACT.** Three approaches to a theory of equisingularity of complex analytic (or algebraic) hypersurfaces are outlined, based respectively on topology (topological equivalence of embedded varieties), differential geometry (Conditions A and B of Whitney) and algebraic geometry (the author's inductive discriminant criterion). For each of these approaches some unsolved questions and (or) conjectures are formulated, especially in regard to the relationship between these three points of view.

# Zariski's question (1971)

---

BULLETIN OF THE  
AMERICAN MATHEMATICAL SOCIETY  
Volume 77, Number 4, July 1971

## SOME OPEN QUESTIONS IN THE THEORY OF SINGULARITIES<sup>1</sup>

BY OSCAR ZARISKI

ABSTRACT. Three approaches to a theory of equisingularity of complex analytic (or algebraic) hypersurfaces are outlined, based respectively on topology (topological equivalence of embedded varieties), differential geometry (Conditions A and B of Whitney) and algebraic geometry (the author's inductive discriminant criterion). For each of these approaches some unsolved questions and (or) conjectures are formulated, especially in regard to the relationship between these three points of view.

*Some open questions in the theory of singularities*, 1971



Oscar Zariski (1899–1986)

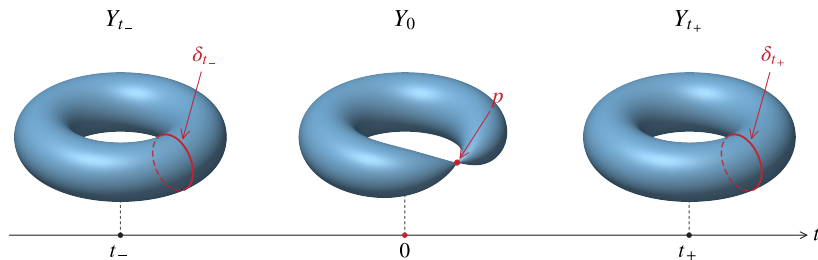
*“it is doubtful whether singularities of complex-analytic varieties are purely topological or even differential-geometric phenomena.”*

## Can a critical point be invisible to the topology of a family?

O. Zariski, *Some open questions in the theory of singularities*, Bull. Amer. Math. Soc. 77 (1971), p. 482.

# A visible degeneration

$Y_s$ : fibre over  $s$ ;  $\delta_s$ : collapsing cycle;  $p$ : critical point.



Must a critical point be topologically visible?

A cycle collapses at the critical value. The construction will use a cusp, not this node.

# Homology and homotopy fibre bundles

---

Let  $f : X \rightarrow B$  be proper; write  $X_b = f^{-1}(b)$ . Choose an open cover  $B = \bigcup U_i$ .

**The condition is required for every  $b \in U_i$ .**

## Integral homology

The fibre inclusion induces

$$H_*(X_b, \mathbb{Z}) \xrightarrow{\sim} H_*(f^{-1}(U_i), \mathbb{Z}).$$

## Homotopy

The fibre inclusion itself is a homotopy equivalence:

$$X_b \hookrightarrow f^{-1}(U_i).$$

homotopy fibre bundle  $\implies \mathbb{Z}$ -homology fibre bundle.

Replacing  $\mathbb{Z}$  by  $\mathbb{Q}$  defines a  $\mathbb{Q}$ -homology fibre bundle.

# The smoothness conjecture

---

**Fernández de Bobadilla–Kollár, 2012**

For a proper morphism  $f : X \rightarrow C$  to a smooth curve, with  $X$  smooth, does topological constancy force smoothness?

$$f \text{ smooth} \implies \text{homotopy fibre bundle} \implies \mathbb{Z}\text{-homology fibre bundle}$$

**The conjectural implications go in the opposite direction.**

We shall disprove both, already over a smooth projective curve.

# Main theorem

---

## Corrêa–Kollár–Schreieder–Wang

There is a morphism  $g : Y \rightarrow \mathbb{P}^1$ , with  $Y$  smooth and projective, such that:

1.  $g$  is a **homotopy fibre bundle**;
2. all fibres are simply connected PL (piecewise-linear) manifolds, pairwise homotopy equivalent;
3.  $g$  is homotopic to a  $C^\infty$  fibre bundle  $\tilde{g} : Y \rightarrow \mathbb{P}^1$  agreeing with  $g$  away from small neighbourhoods of the singular fibres;
4.  $g$  is **not smooth**, and is not a locally trivial  $C^0$  (topological) bundle.

**The smallest construction has  $\dim_{\mathbb{C}} Y = 5$  and  $\dim_{\mathbb{C}} \text{Crit}(g) = 1$ .**

Here  $\text{Crit}(g) = \{y \in Y : dg_y = 0\}$  is the critical locus.

# Two operations in the construction

---

## 1. Hide the local cohomology

$$f : \mathbb{C}^2 \rightarrow \mathbb{C}, \quad f(x, y) = x^2 + y^3.$$

$$M_t = f^{-1}(t) \cap B_\varepsilon, \quad 0 < |t| \ll \varepsilon.$$

$B_\varepsilon \subset \mathbb{C}^2$ : a small ball about 0.

$h : M_t \rightarrow M_t$ : return map around 0.

$$T = h^* : H^1(M_t, \mathbb{Z}) \longrightarrow H^1(M_t, \mathbb{Z}).$$

$\det(T - I) = 1$ , where  $I = \text{id}$ .

free diagonal quotient

---

integral homology constancy

## 2. Remove the homotopy obstruction

embed the quotient family

blow up an ambient family  
with simply connected fibres

apply Whitehead's theorem

---

homotopy constancy

**The critical points survive both operations.**

# The local data: a Milnor fibre

Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be holomorphic, with an isolated critical point and  $n \geq 2$ . Write  $\Delta$  for a small disc about 0 and  $\Delta^* = \Delta \setminus \{0\}$ .

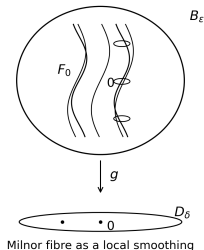
Choose a small ball  $B_\varepsilon \subset \mathbb{C}^n$  about 0 and  $0 < |t| \ll \varepsilon$ .

$$M_t = f^{-1}(t) \cap B_\varepsilon.$$

Write  $M = M_t$  for one chosen nearby fibre. The reduced cohomology is concentrated in degree  $n - 1$ :

$$\tilde{H}^{n-1}(M, \mathbb{Z}) \cong \mathbb{Z}^{\mu(f)}.$$

$\tilde{H}$  denotes reduced cohomology; its rank  $\mu(f)$  is the *Milnor number*, counting vanishing classes.



In the figure,  $g = f$ ,  $D_\delta = \Delta$ , and  $F_0 = f^{-1}(0) \cap B_\varepsilon$  is the central local fibre.

# Transport around the critical value

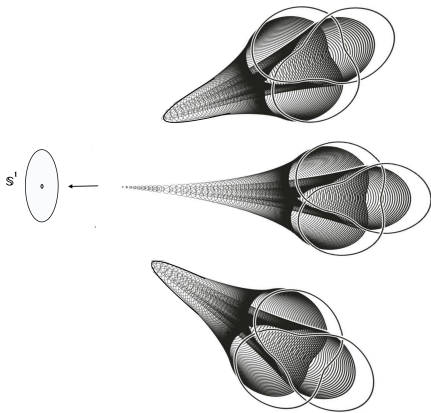
## Over the punctured parameter disc

For the local map  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ , the smooth fibres are transported along paths in  $\Delta^* = \Delta \setminus \{0\}$ .

$$M_t = f^{-1}(t) \cap B_\varepsilon, \quad h : M_t \rightarrow M_t.$$

Here  $h$  is the return map after one turn around 0; the pictured  $S^1$  parametrises that turn.

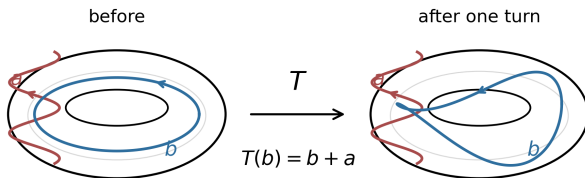
This transports the local smoothing; it is not the later diagonal quotient.



Schematic illustration of the Milnor fibration.

# Monodromy: the nodal illustration

Let  $F_t$  be a nearby smooth genus-one fibre. One positive turn around 0 gives a return map  $h : F_t \rightarrow F_t$ .



The picture writes  $T$  for  $h_* : H_1(F_t, \mathbb{Z}) \rightarrow H_1(F_t, \mathbb{Z})$ ;  $a, b \in H_1(F_t, \mathbb{Z})$  are the pictured cycle classes.

$$h_*(a) = a, \quad h_*(b) = b + a.$$

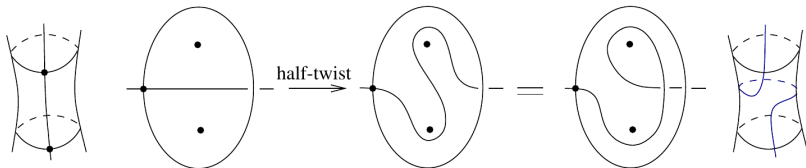
**The vanishing cycle  $a$  is fixed.** The cycle  $b$  crosses it.

Nodal Dehn-twist schematic; orientations determine the sign. Outside this picture,  $T$  denotes the cohomological action for the cusp.

# A half-twist downstairs, a Dehn twist upstairs

**The nodal model**  $z_1^2 + z_2^2$

A suitable projection describes the local fibre as a double cover with two branch points.



A half-twist interchanging the branch points lifts to a Dehn twist about the vanishing cycle.

**The picture illustrates the return map, not the collapse itself.**

Illustration after Auroux–Smith. Nodal monodromy, not the order-six monodromy of the cusp.

# Why the cusp is the right singularity

---

## A finite-order geometric symmetry

$$f(x, y) = x^2 + y^3, \quad \mu(f) = 2.$$

For  $\zeta = e^{2\pi i/6}$ , define

$$\gamma : \mathbb{C}^2 \longrightarrow \mathbb{C}^2,$$

$$(x, y) \longmapsto (\zeta^3 x, \zeta^2 y).$$

It preserves  $f$ . Its restriction  $\gamma|_M : M \rightarrow M$  represents the weighted-homogeneous monodromy.

## The action on $H^1(M, \mathbb{Z}) \cong \mathbb{Z}^2$

Write  $T = (\gamma|_M)^*$ . In an integral basis,

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Its characteristic polynomial is

$$\chi_T(\lambda) := \det(\lambda I - T) = \lambda^2 - \lambda + 1.$$

$$T - I = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \det(T - I) = 1.$$

**$T - I$  is invertible over  $\mathbb{Z}$ .**

# Suspension and holonomy

Let  $F$  be a smooth manifold,  $x \in F$ , and  $\gamma : F \rightarrow F$  a diffeomorphism.

## The mapping torus

$$M_\gamma = \frac{F \times [0, 1]}{(x, 1) \sim (\gamma(x), 0)}.$$

Its bundle projection is

$$\begin{aligned} p_\gamma : M_\gamma &\longrightarrow S^1, \\ [x, s] &\longmapsto e^{2\pi i s}. \end{aligned}$$

One circuit returns  $x$  to  $\gamma(x)$ . The same return map is the transverse *holonomy* of the suspension foliation.



schematic fibre; horizontal coordinate is the interval parameter

# The free diagonal quotient

Take a projective degeneration  $\pi : Z \rightarrow \Delta$ , with  $Z$  smooth and isolated critical points over 0. Let  $\gamma : Z \rightarrow Z$  have finite order  $m$ , preserve  $\pi$  and fix the critical points.

On an abelian variety  $A$ , choose  $\tau : A \rightarrow A$ ,  $a \mapsto a + a_0$ , with  $a_0$  of exact order  $m$ .

$$\begin{array}{ccc} A \times Z & \xrightarrow{\text{finite étale}} & Q = (A \times Z) / \langle (\tau, \gamma) \rangle \\ \text{pr}_A \downarrow & & \downarrow q \\ A & \xrightarrow{\text{translation quotient}} & A' = A / \langle \tau \rangle \end{array}$$

The diagonal action is  $(a, z) \mapsto (\tau(a), \gamma(z))$ . The induced maps are

$$\pi_Q : Q \rightarrow \Delta, \quad [a, z] \mapsto \pi(z), \quad q : Q \rightarrow A', \quad [a, z] \mapsto [a].$$

**Translation makes the action free; critical points of  $\pi_Q$  remain.**

# The diagonal quotient theorem

---

Let  $\pi : Z \rightarrow \Delta$  be projective, with  $Z$  smooth of dimension  $n$  and a finite nonempty set of critical points  $p_i$  over 0.

Let  $\gamma : Z \rightarrow Z$  be a finite-order symmetry preserving  $\pi$  and fixing each  $p_i$ . Let  $M_i$  be its invariant Milnor fibre at  $p_i$ , and set

$$T_i = (\gamma|_{M_i})^* : H^{n-1}(M_i, \mathbb{Z}) \longrightarrow H^{n-1}(M_i, \mathbb{Z}).$$

Assume  $\chi_{T_i}(1) = \pm 1$  for every  $i$ , where  $\chi_{T_i}(\lambda) = \det(\lambda I - T_i)$ . For an abelian variety  $A$  with a translation  $\tau : A \rightarrow A$  of the same order,

$$(A \times Z) / \langle (\tau, \gamma) \rangle \longrightarrow \Delta$$

is a  $\mathbb{Z}$ -homology fibre bundle, but not a homotopy fibre bundle.

# The integral quotient lemma

---

Let  $n \geq 2$  and suppose the only nonzero integral cohomology of  $M$  is

$$H^0(M, \mathbb{Z}) = \mathbb{Z}, \quad H^{n-1}(M, \mathbb{Z}) = \mathbb{Z}^r.$$

Let  $\gamma : M \rightarrow M$  have finite order  $m$  and put

$$T = \gamma^* : H^{n-1}(M, \mathbb{Z}) \longrightarrow H^{n-1}(M, \mathbb{Z}).$$

Choose an abelian variety  $A$  with a translation  $\tau : A \rightarrow A$  of order  $m$ . Then

$$q_M : \frac{A \times M}{\langle (\tau, \gamma) \rangle} \longrightarrow A / \langle \tau \rangle, \quad [a, x] \longmapsto [a],$$

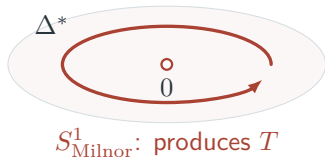
induces an isomorphism on integral cohomology if and only if

$$\chi_T(1) = \pm 1, \quad \chi_T(\lambda) = \det(\lambda I - T).$$

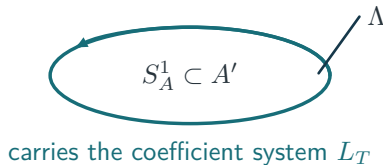
# Two different circles

In our weighted-homogeneous examples, put  $\Lambda = \tilde{H}^{n-1}(M, \mathbb{Z})$ . The chosen symmetry induces the Milnor action  $T : \Lambda \rightarrow \Lambda$ .

## Parameter space



## Abelian quotient

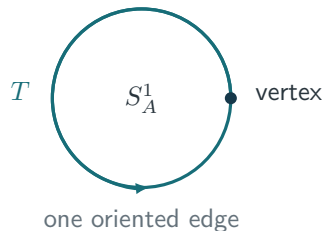


$S^1_{\text{Milnor}}$  is a small circle around  $0$  in  $\Delta^*$ ;  $S^1_A$  is the circle factor of  $A'$  carrying the twisting.  $L_T$  is the local system of lattices with fibre  $\Lambda$  and transport  $T$  in the chosen orientation of  $S^1_A$ .

**One circle defines  $T$ ; the other carries the cohomology calculation.**

# The calculation on a circle

Let  $L_T$  be the local system on  $S_A^1$  with fibre  $\Lambda = \tilde{H}^{n-1}(M, \mathbb{Z})$  and monodromy  $T : \Lambda \rightarrow \Lambda$ .



## One vertex and one oriented edge

The cellular cochain complex is

$$0 \longrightarrow \Lambda \xrightarrow{T-I} \Lambda \longrightarrow 0,$$

with the two copies of  $\Lambda$  in degrees 0 and 1.

$$\begin{aligned} H^0(S_A^1, L_T) &= \ker(T - I), \\ H^1(S_A^1, L_T) &= \operatorname{coker}(T - I). \end{aligned}$$

# Why integral coefficients matter

---

## The cusp: complete cancellation

$$\det(T - I) = 1.$$

Both kernel and cokernel vanish:

$$H^*(S^1, L_T) = 0 \quad \text{over } \mathbb{Z}.$$

## No eigenvalue 1: not enough

For  $\Lambda = \mathbb{Z}$  and  $T : \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $v \mapsto -v$ , the differential is multiplication by  $-2$ .

$$H^0(S^1, L_T) = 0, \quad H^1(S^1, L_T) = \mathbb{Z}/2\mathbb{Z}.$$

Rational vanishing leaves integral torsion.

**The needed condition is  $\det(T - I) = \pm 1$ , not merely  $\det(T - I) \neq 0$ .**

# From the circle to the quotient

---

For the chosen Milnor fibre  $M$ , write

$$\mathcal{E}_M = (A \times M) / \langle (\tau, \gamma) \rangle, \quad q_M : \mathcal{E}_M \longrightarrow A' := A / \langle \tau \rangle, \quad [a, x] \mapsto [a].$$

## Topological reduction

Put the translation in one circle factor of  $A$ .

The remaining torus factors are untwisted.

Let  $p_T : E_T \rightarrow S_A^1$  be the restriction of  $q_M$

to this circle, with fibre  $M$ .

The second row is acyclic when  $\det(T - I) = \pm 1$ . Leray and Künneth give

$$q_M^* : H^*(A', \mathbb{Z}) \xrightarrow{\sim} H^*(\mathcal{E}_M, \mathbb{Z}).$$

## Only two cohomology rows

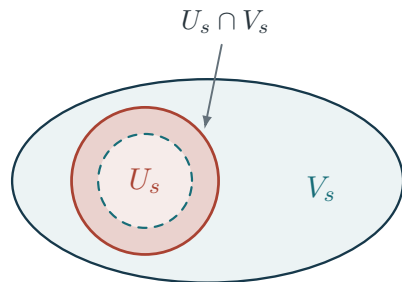
The fibre-cohomology sheaves are

$$R^0(p_T)_* \mathbb{Z} = \mathbb{Z}, \quad R^{n-1}(p_T)_* \mathbb{Z} = L_T.$$

Here  $R^j(p_T)_* \mathbb{Z}$  has stalk  $H^j(M, \mathbb{Z})$ , with transport induced by the bundle.

# The local comparison becomes global

Put  $Q_s = \pi_Q^{-1}(s)$ , for  $s = 0, t$ . Choose  $Q_s = U_s \cup V_s$ , with  $U_s$  the quotient Milnor regions and  $V_s$  the regular complement, overlapping along their collars.



$$Q_s = U_s \cup V_s, \quad s = 0, t$$

## Compare central and nearby fibres

The quotient lemma compares the integral cohomology of  $U_0$  and  $U_t$ . Ehresmann

identifies  $V_0, V_t$  and the overlaps  $U_s \cap V_s$ . These comparisons respect the maps in the Mayer–Vietoris sequences.

**Mayer–Vietoris gives a  $\mathbb{Z}$ -homology fibre bundle.**

# The natural Mayer–Vietoris comparison

For  $Q_s = \pi_Q^{-1}(s) = U_s \cup V_s$ ,  $s = 0, t$ , set

$$A_s^j = H^j(U_s, \mathbb{Z}) \oplus H^j(V_s, \mathbb{Z}), \quad B_s^j = H^j(U_s \cap V_s, \mathbb{Z}).$$

$$\begin{array}{ccccccccc}
 A_0^{k-1} & \longrightarrow & B_0^{k-1} & \longrightarrow & H^k(Q_0, \mathbb{Z}) & \longrightarrow & A_0^k & \longrightarrow & B_0^k \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 A_t^{k-1} & \longrightarrow & B_t^{k-1} & \longrightarrow & H^k(Q_t, \mathbb{Z}) & \longrightarrow & A_t^k & \longrightarrow & B_t^k
 \end{array}$$

Restricting a compatible retraction of the total neighbourhood gives  $r_t : Q_t \rightarrow Q_0$ . Its pullbacks induce the vertical maps. The outer four are isomorphisms; the five lemma gives the middle one.

# Why this is not yet a homotopy fibre bundle

---

A finite étale cover detects what the quotient hides

$$A \times Z \longrightarrow Q = (A \times Z) / \langle (\tau, \gamma) \rangle.$$

If  $Q \rightarrow \Delta$  were homotopically constant, the covering family would be homotopically constant too. But  $Z \rightarrow \Delta$  has isolated singularities with nonzero vanishing cohomology.

Künneth retains it after taking the product with  $A$ .

**The quotient is a  $\mathbb{Z}$ -homology fibre bundle, but not a homotopy fibre bundle.**

# A concrete family of bielliptic surfaces

---

In homogeneous coordinates  $[x : y : z]$  on  $\mathbb{P}^2$ , take

$$S = \{zy^2 + x^3 + tz^3 = 0\} \subset \mathbb{P}^2 \times \Delta, \quad \zeta = e^{2\pi i/6}.$$

Projection to  $t$  gives  $S \rightarrow \Delta$ . The order-six symmetry  $\gamma : S \rightarrow S$  is

$$\gamma([x : y : z], t) = ([\zeta^2 x : \zeta^3 y : z], t).$$

Choose an elliptic curve  $A$  and an order-six translation  $\tau : A \rightarrow A$ .

$$Q = (S \times A) / \langle (\gamma, \tau) \rangle \longrightarrow \Delta.$$

General fibres are bielliptic surfaces. The central fibre is singular, but integral homology is locally constant.

# Beyond the cusp: Brieskorn–Pham examples

---

For the local map  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  given by

$$f = x_1^{c_1} + \cdots + x_n^{c_n}, \quad c_i \geq 2, \quad \gcd(c_i, c_j) = 1 \ (i \neq j),$$

let  $M$  be its Milnor fibre and  $T : \tilde{H}^{n-1}(M, \mathbb{Z}) \rightarrow \tilde{H}^{n-1}(M, \mathbb{Z})$  its monodromy. Pham's calculation gives  $\chi_T(1) = \pm 1$ .

For  $x^2 + y^3 + z^5$ , the Milnor number is 8. The Milnor lattice is of type  $E_8$ , and the monodromy is related to its Coxeter transformation.

Adding an even number of squares to the cusp preserves its monodromy action on reduced middle cohomology, with a shift of degree.

# Globalisation: six cuspidal fibres

Take  $L = \mathcal{O}_{\mathbb{P}^1}(1)$  and  $s \in H^0(\mathbb{P}^1, L^{\otimes 6})$  with six simple zeros  $t_1, \dots, t_6$ .

$$S = \{zy^2 + x^3 + sz^3 = 0\} \subset \mathbb{P}(\mathcal{O} \oplus L^{\otimes 2} \oplus L^{\otimes 3}).$$

$z, x, y$  correspond to  $\mathcal{O}, L^2, L^3$ , respectively.



Each marked fibre is a cusp (Kodaira type II). The symmetry extends globally. Keeping  $A$  and  $\tau$  from the local example, the smooth threefold

$$h : V = (S \times A) / \langle (\gamma, \tau) \rangle \longrightarrow \mathbb{P}^1$$

is a  $\mathbb{Z}$ -homology fibre bundle with six singular fibres.

# A birational operation changes the homotopy problem

Embed  $V$  over  $\mathbb{P}^1$  in a smooth family  $p : P \rightarrow \mathbb{P}^1$  with simply connected fibres and  $c = \operatorname{codim}_P V \geq 2$ .

Let  $\beta : Y = \operatorname{Bl}_V P \rightarrow P$  be the blow-up map and set  $g = p \circ \beta$ .

$$\begin{array}{ccccc} E = \mathbb{P}(N_{V/P}) & \xrightarrow{\quad\quad\quad} & Y = \operatorname{Bl}_V P & & \\ \downarrow \mathbb{P}^{c-1} & & \downarrow \beta & \searrow g & \\ V & \xrightarrow{\text{smooth centre}} & P & \nearrow p & \mathbb{P}^1 \end{array}$$

$N_{V/P}$  is the normal bundle;  $E = \beta^{-1}(V) = \mathbb{P}(N_{V/P})$  is the exceptional divisor. In the smallest construction,  $P_t = p^{-1}(t)$  is rational,  $\dim_{\mathbb{C}} P_t = 4$ , and  $\dim_{\mathbb{C}} Y = 5$ .

# The ambient geometry in dimension five

---

## Sketch of Example 19: rational four-dimensional fibres

The explicit construction uses a degree-six del Pezzo surface: the blow-up of  $\mathbb{P}^2$  at its three coordinate vertices, with compatible involution and order-three symmetry.

The finite quotient is smooth along the embedded centre  $V$ . A suitable resolution away from  $V$  gives the smooth ambient family  $P \rightarrow \mathbb{P}^1$ .

The ambient fibres are rational and simply connected. Blowing up  $V$  gives the five-dimensional total space  $Y$ .

# Apply the blow-up formula over open sets

---

For  $p : P \rightarrow \mathbb{P}^1$  and  $c = \operatorname{codim}_P V$ , let  $U \subset \mathbb{P}^1$  be small. Put  $P_U = p^{-1}(U)$  and  $V_U = V \cap P_U$ . Both total spaces are smooth;  $g = p \circ \beta$ , with  $\beta$  the blow-up map.

$$H^k(\operatorname{Bl}_{V_U} P_U, \mathbb{Z}) \cong H^k(P_U, \mathbb{Z}) \oplus \bigoplus_{j=1}^{c-1} H^{k-2j}(V_U, \mathbb{Z}).$$

Write  $Y_U = g^{-1}(U)$  and  $E_U = \beta^{-1}(V_U)$ . The maps use pullback by  $\beta_U : Y_U \rightarrow P_U$  and the Gysin (cohomological pushforward) map

$$(i_U)_* : H^{k-2}(E_U, \mathbb{Z}) \longrightarrow H^k(Y_U, \mathbb{Z}), \quad i_U : E_U \hookrightarrow Y_U.$$

They commute with restriction in  $U$ . After sheafification,

$$R^k g_* \mathbb{Z} \cong R^k p_* \mathbb{Z} \oplus \bigoplus_{j=1}^{c-1} R^{k-2j} (p|_V)_* \mathbb{Z}.$$

# The relative blow-up formula

---

For  $h = p|_V : V \rightarrow \mathbb{P}^1$  and  $g = p \circ \beta : Y \rightarrow \mathbb{P}^1$ , naturality gives

$$R^k g_* \mathbb{Z} \cong R^k p_* \mathbb{Z} \oplus \bigoplus_{j=1}^{c-1} R^{k-2j} h_* \mathbb{Z}.$$

These are sheaves of fibre cohomology: for example,  $(R^k g_* \mathbb{Z})_t = H^k(Y_t, \mathbb{Z})$ , where  $Y_t = g^{-1}(t)$ .

## Cohomology

The ambient terms are locally constant by smoothness; the centre terms by the quotient construction.

## Critical points

The blow-up proposition also gives

$$g \text{ smooth} \iff h \text{ smooth}.$$

**Thus  $g$  is a  $\mathbb{Z}$ -homology fibre bundle and is not smooth.**

# Simple connectivity and Whitehead

---

## First control the fundamental group

For a small disc  $U \subset \mathbb{P}^1$  about  $b$ , put  $P_U = p^{-1}(U)$ . This smooth ambient space is simply connected. Blowing up the smooth centre in codimension at least two preserves the fundamental group.

$$\pi_1(g^{-1}(U)) = 0.$$

The fibre  $Y_b = g^{-1}(b)$  is a retract of  $g^{-1}(U)$ , so it is simply connected too.

## Then use integral homology

For every  $t \in U$ , the inclusion

$$j_t : Y_t = g^{-1}(t) \hookrightarrow g^{-1}(U)$$

is an integral homology equivalence between simply connected spaces of CW type (homotopy type of cell complexes). **Whitehead makes  $j_t$  a homotopy equivalence.**

# The local blow-up chart

---

Locally  $p : \mathbb{C}^5 \rightarrow \mathbb{C}$  sends  $(x, y, u, v, t)$  to  $t$ . The smooth centre  $V$  has ideal  $I_V$ :

$$V = \{u = 0, t = x^2 + y^3\}, \quad I_V = (u, t - x^2 - y^3).$$

With affine blow-up coordinate  $w$ , the chart  $t - x^2 - y^3 = uw$  gives

$$g : \mathbb{C}^5 \longrightarrow \mathbb{C}, \quad (x, y, u, v, w) \longmapsto x^2 + y^3 + uw.$$

A linear change of the quadratic variables gives

$$g = x_1^2 + x_2^2 + x_3^2 + x_4^3, \quad x_5 \text{ free.}$$

**The critical locus is one-dimensional; the transverse Milnor number is 2.**

# The link and the Milnor fibre are different spaces

---

Let  $f : \mathbb{C}^4 \rightarrow \mathbb{C}$  be  $f = x_1^2 + x_2^2 + x_3^2 + x_4^3$ . Write  $B_\varepsilon^8 \subset \mathbb{C}^4$  for a small real 8-ball and  $S_\varepsilon^7 = \partial B_\varepsilon^8$ .

## The central transverse germ

Its *link* is

$$L = f^{-1}(0) \cap S_\varepsilon^7.$$

Brieskorn–Milnor give PL-homeomorphisms

$$L \cong_{\text{PL}} S^5, \quad \text{Cone}(L) \cong_{\text{PL}} B^6.$$

$\text{Cone}(L)$  is the cone on  $L$ ;  $B^6$  is a real 6-ball.

The free complex coordinate then gives a PL chart of the fibre.

## A nearby transverse level

For  $0 < |t| \ll \varepsilon$ , its Milnor fibre is

$$M_t = f^{-1}(t) \cap B_\varepsilon^8.$$

The vanishing cohomology survives:

$$\tilde{H}^3(M_t, \mathbb{Z}) = \mathbb{Z}^2.$$

It obstructs local  $C^0$ -triviality of  $g$ .

**The fibres are PL-manifolds; the morphism is not a topological fibre bundle.**

# The algebraic map and its smooth representative

---

## The algebraic map

$$g : Y \longrightarrow \mathbb{P}^1$$

has critical points and is not locally  $C^0$ -trivial.

Local  $C^0$ -triviality would mean a homeomorphism  $g^{-1}(U) \cong Y_b \times U$  over  $U$ , for  $b \in U$ .

## A different map on the same manifold

$$\tilde{g} : Y \longrightarrow \mathbb{P}^1$$

is a proper  $C^\infty$ -submersion, hence a smooth fibre bundle.

$$g \simeq \tilde{g}, \quad g = \tilde{g} \text{ outside small neighbourhoods of the singular fibres.}$$

**This changes the map, not the variety  $Y$ .**

# Smooth representative: the precise statement

---

Let  $g : X \rightarrow \Sigma$  be a proper holomorphic map of connected complex manifolds, with

$$\dim_{\mathbb{C}} \Sigma = 1, \quad \dim_{\mathbb{C}} X \geq 4.$$

Assume that every  $R^i g_* \mathbb{Z}$  is a local system and that every fibre is simply connected.

For any analytic neighbourhood  $U$  of the critical values, there is a proper  $C^\infty$ -submersion

$$\tilde{g} : X \rightarrow \Sigma$$

homotopic to  $g$  and equal to  $g$  outside  $g^{-1}(U)$ .

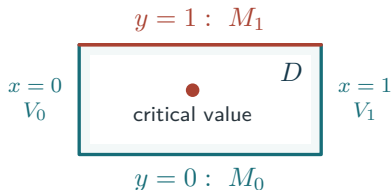
**Apply this to the constructed map  $g : Y \rightarrow \mathbb{P}^1$ .**

# Smale: a product of the total space

Choose a coordinate square  $D = I_x \times I_y \subset \mathbb{P}^1$ , with  $I_x = I_y = [0, 1]$ , containing one critical value in its interior. Put  $W = g^{-1}(D)$  and fix a smooth fibre  $F = g^{-1}(b)$ ,  $b \in \partial D$ .

## The boundary faces of $W$

$$\begin{aligned} M_i &= g^{-1}(I_x \times \{i\}), \\ V_j &= g^{-1}(\{j\} \times I_y), \quad i, j = 0, 1. \end{aligned}$$



Labels denote faces in  $W$ , not fibres.

## The product need not respect $g$ on the top face $M_1$ .

Ehresmann supplies the boundary trivialisations. CKSW, Proposition 14, Steps 1–2, and Theorem 15.

## Why an $h$ -cobordism?

$W$  is a cobordism from  $M_0$  to  $M_1$ . Homology constancy, simple connectivity and Whitehead give

$$M_i \hookrightarrow W \quad \text{homotopy equivalences,} \quad i = 0, 1.$$

This is the  $h$ -cobordism condition.

## Smale: relative $h$ -cobordism

For simply connected  $W$ ,  $\dim_{\mathbb{R}} W \geq 8$ , the bottom and side trivialisations extend to

$$\Psi : F \times D \xrightarrow{\cong} W,$$

a diffeomorphism of total spaces.

# Cerf: a path of pseudo-isotopies

---

Write  $\Psi_{\text{top}} : F \times I_x \rightarrow M_1$  for the top restriction of  $\Psi$ . Choose an Ehresmann trivialisation  $\theta : F \times I_x \rightarrow M_1$  with  $g(\theta(z, x)) = (x, 1)$ , agreeing with  $\Psi_{\text{top}}$  at  $x = 0$ .

## The remaining discrepancy

$$\begin{aligned}\Phi &= \theta^{-1} \circ \Psi_{\text{top}}, \\ \Phi : F \times I_x &\xrightarrow{\cong} F \times I_x, \\ \Phi|_{F \times \{0\}} &= \text{id}.\end{aligned}$$

Such a diffeomorphism is a *pseudo-isotopy*; it need not preserve  $x$ .

## The remaining task is explicit.

Use the path  $\Phi_s$  to correct the product at the top face, without losing compatibility with the base on the other faces.

The next step constructs that correction; only then do we define the new submersion.

CKSW, Proposition 14, Steps 3–4. Cerf (1970), pp. 7–8, Définition 2 and Théorème 0.

## Cerf's pseudo-isotopy theorem

For compact, simply connected  $F$  without boundary and  $\dim_{\mathbb{R}} F \geq 6$ , Cerf gives a smooth path

$$\begin{aligned}\Phi_s : F \times I_x &\xrightarrow{\cong} F \times I_x, \\ 0 \leq s \leq 1, \quad \Phi_0 &= \Phi, \quad \Phi_1 = \text{id},\end{aligned}$$

through pseudo-isotopies.

# Boundary correction: the actual diffeomorphism

---

Keep  $D = I_x \times I_y$ ,  $W = g^{-1}(D)$  and the products  $\Psi, \theta$  from the preceding two slides. Cerf gives pseudo-isotopies  $\Phi_s : F \times I_x \rightarrow F \times I_x$  with

$$\Phi_0 = \theta^{-1} \circ \Psi_{\text{top}}, \quad \Phi_1 = \text{id}, \quad \Phi_s|_{F \times \{0\}} = \text{id}.$$

Use the path in the  $y$ -direction

$$\Xi : F \times D \longrightarrow F \times D, \quad \Xi(z, x, y) = (\Phi_{1-y}^{-1}(z, x), y),$$

$$\widehat{\Psi} = \Psi \circ \Xi : F \times D \xrightarrow{\cong} W.$$

**Bottom and left: unchanged**

$$\widehat{\Psi}(z, x, 0) = \Psi(z, x, 0),$$

$$\widehat{\Psi}(z, 0, y) = \Psi(z, 0, y).$$

**Top and right: over the base**

$$\widehat{\Psi}_{\text{top}} = \theta,$$

$$g(\widehat{\Psi}(z, 1, y)) = (1, y).$$

**The right face stays fibre-preserving; its chosen trivialisation need not stay fixed.**

# From the boundary to a global submersion

---

Write  $\text{pr}_D : F \times D \rightarrow D$ ,  $(z, x, y) \mapsto (x, y)$ . The corrected product  $\widehat{\Psi} : F \times D \rightarrow W$  satisfies  $g \circ \widehat{\Psi} = \text{pr}_D$  on  $F \times \partial D$ . An adjustment in a collar (a small boundary neighbourhood) gives a product  $\Psi'$  for which this equality holds *near*  $F \times \partial D$ .

## Define the new map

$$\widetilde{g}_D : W \longrightarrow D, \quad \widetilde{g}_D = \text{pr}_D \circ (\Psi')^{-1}.$$

This is the local restriction  $\widetilde{g}|_W$ ; it is a submersion and equals  $g$  near  $\partial W$ .

## Write the homotopy and glue

$$H : W \times [0, 1] \longrightarrow D, \quad H(w, s) = (1 - s)g(w) + s\widetilde{g}_D(w).$$

This formula uses the chosen real coordinates on the convex square  $D$ . It is constant near  $\partial W$ . Repeat near each critical value and glue to  $g$  outside. This gives a proper submersion  $\widetilde{g} : Y \rightarrow \mathbb{P}^1$  and a homotopy from  $g$  to  $\widetilde{g}$ .

# Application: a finite-type universal cover

---

Let  $\rho : E \rightarrow \mathbb{P}^1$  be a double cover from an elliptic curve, unramified over the critical values of  $g$ .

$$\begin{array}{ccc} X = Y \times_{\mathbb{P}^1} E & \xrightarrow{\quad} & Y \\ f \downarrow & & \downarrow g \\ E & \xrightarrow{\quad \rho \quad} & \mathbb{P}^1 \end{array}$$

Here  $f : X \rightarrow E$  is the projection  $(y, e) \mapsto e$ . Since  $f$  is a homotopy fibre bundle with simply connected fibres,  $\pi_1(X) \cong \pi_1(E) \cong \mathbb{Z}^2$ .

Let  $u : \mathbb{C} \rightarrow E$  be the universal covering map. Then  $\tilde{X} = X \times_E \mathbb{C}$  is the universal cover of  $X$  and has the homotopy type of a compact fibre.

**The universal cover has the homotopy type of a finite cell complex, but  $f$  is not smooth.**

# Application: real and holomorphic one-forms

---

The Albanese map  $f : X \rightarrow E$  has critical points but is homotopic to a smooth submersion  $\tilde{f} : X \rightarrow E$ . Choose a basis  $dx, dy$  of translation-invariant real one-forms on  $E$ .

## Closed real one-forms

Every  $0 \neq u \in H^1(X, \mathbb{R})$  has a representative

$$\alpha = \tilde{f}^*(a dx + b dy),$$

$$a, b \in \mathbb{R}, \quad (a, b) \neq (0, 0),$$

which is closed and nowhere zero.

**A nowhere-zero closed real one-form need not imply a nowhere-zero holomorphic one-form.**

This is the negative answer to Kotschick's question.

## Holomorphic one-forms

Every  $\omega \in H^0(X, \Omega_X^1)$  comes from a form  $\eta \in H^0(E, \Omega_E^1)$ :

$$\omega = f^*\eta.$$

It vanishes at the critical points of  $f$ .

# Closed forms and harmonic representatives

---

## Harmonic representatives

Although every nonzero real class on the example has a nowhere-zero closed representative, its harmonic representative, for a Kähler metric, has zeros: holomorphic one-forms are pulled back by the singular Albanese map.

## Two maps, one homotopy class

The closed real representatives use  $\tilde{f}^*$ ; the holomorphic forms use  $f^*$ .

# A boundary in low dimension

---

## Threefold rigidity

Let  $Y$  be a smooth complex projective threefold and  $g : Y \rightarrow \mathbb{P}^1$  a  $\mathbb{Q}$ -homology fibre bundle with simply connected fibres. Then  $g$  is smooth. The proof combines elliptic base change, the decomposition theorem and results of Hao–Schreieder on holomorphic one-forms.

|                                 |                           |                           |
|---------------------------------|---------------------------|---------------------------|
| $\dim_{\mathbb{C}} Y = 3$       | $\dim_{\mathbb{C}} Y = 4$ | $\dim_{\mathbb{C}} Y = 5$ |
| rigidity under these hypotheses | open in the paper         | explicit counterexample   |

# Questions suggested by the construction

---

## Dimension four

Can the homotopy counterexample be constructed with  $\dim_{\mathbb{C}} Y = 4$ ?

## Stronger fibre constancy

Can the fibres be pairwise PL-homeomorphic, not merely homotopy equivalent?

## Canonical positivity

Can every fibre have nef canonical class?

## General type

The present construction does not answer every positivity question. What happens for varieties of general type carrying a nowhere-zero closed real one-form?

# References

---

## Main mathematical sources

M. Corrêa, J. Kollár, S. Schreieder and B. Wang,

*Invisible singularities in complex algebraic geometry*, arXiv:2608.10973.

J. Fernández de Bobadilla and J. Kollár, *Homotopically trivial deformations*, J. Singul. 5 (2012), 85–93.

O. Zariski, *Some open questions in the theory of singularities*, Bull. AMS 77 (1971), 481–491.

J. Milnor, *Singular Points of Complex Hypersurfaces*, 1968.

## Further ingredients

Brieskorn's description of the links, Smale's relative  $h$ -cobordism theorem and Cerf's pseudo-isotopy theorem; see the references in the main paper.

# Invisible globally, nontrivial locally

---

**The local cycles do not disappear.**

The diagonal quotient makes their coefficient-system cohomology vanish.

**The blow-up supplies simple connectivity.**

Integral homology constancy then becomes homotopy constancy.

**The map still remembers its singularities.**

It is not a  $C^0$ -bundle, even though its homotopy class contains a smooth one.

Thank you.